THE DEVELOPMENT OF FUNCTIONS ASSOCIATED WITH SURFACE WAVES OVER AN INCLINED BOTTOM

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TECHNICAL REPORT NO. 26

SEPTEMBER 17, 1954

PREPARED UNDER CONTRACT Nonr-225(11)
(NR-041-086)
FOR
OFFICE OF NAVAL RESEARCH

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Introduction.

The problem of surface waves over an inclined bottom can be formulated thus: construct the harmonic functions $\varphi(x,y)$, the velocity potential of the flow, which on the surface $y = 0$, $x > 0$, satisfy the boundary condition $\frac{\partial\varphi}{\partial y} = \varphi$, while on the "bottom" $x = y \cot \alpha\pi$, $y < 0$ the normal derivative of φ vanishes. Here it is appropriate to admit for α any number between 0 and 1, and even the limit case $\alpha = 1$ in order to include the so-called dock problem. The domain of regularity of φ is the sector included between the two rays formed by surface and bottom.

If instead of φ the analytic function $f(z)$ of the complex variable, $z = x + iy$, is considered whose real part is $\varphi(x,y)$ and a proper choice is made of the arbitrary additive constant, the function $f(z)$ becomes solution of the difference-differential equation in a sector of double angle $2\pi\alpha$

$$\text{(E)} \qquad (d/dz)\,(f(z) - f(\varepsilon z)) + i(f(z) + f(\varepsilon z)) = 0, \qquad \varepsilon = e^{-2\pi i\alpha},$$

and this equation permits the analytic extension of f into the logarithmic Riemann surface. The relationship of f and φ is not reversible, the class of real parts of such functions f being larger than that of functions φ as defined above. Nevertheless, the versatility provided by the theory of regular functions of a complex variable justifies, even in the consideration of surface waves, the concentration of interest on the solution of (E).

The first problem, and the one first solved, was that of the construction of the standing wave in the infinite sector $0 \geq \arg z \geq -\pi\alpha$ whose associated velocity remains bounded throughout the sector [1], [3], [5]. The corresponding complex function $I_o(z)$ has been the subject of closer study in [2]. The reason is that from it other solutions of (E) can be derived by simple integral formulae, and that the manifold of these is wide enough to provide a basis for all solutions of (E) which remain regular and continuous in the neighborhood $|z| < r$ for bounded $|\arg z|$, no matter how small $r > 0$. This is one of the main results of [2] (Theorem 12.1). A similar question now arises as to solutions of (E) which tend to zero as $z \to \infty$ in the sector $0 \geq \arg z > -2\pi\alpha$. Indeed we may compare the situation with that arising in the case of analytic functions regular and single valued in the whole plane excepting possibly $z = 0$ and $z = \infty$. Here the Laurent series development yields the dissection of such functions into a sum of two, the first continuous at the origin, the second tending to zero at ∞. But this Laurent decomposition exists not only for the indicated class of functions, but for functions which need be regular only in a circular ring, and indeed only exist on a single circumference with origin as center. For then the Laurent series is the Fourier decomposition of the function on this circumference. Now there reigns a perfect analogy in the class of functions, solutions of (E), and this paper is devoted to its establishment.

Throughout this paper repeated use is made of the results of [2] and for this reason we have adhered to the same notations as in [2].

1. <u>Notations and recall of basic facts</u>.

Denote by $I_o(z)$ Isaacson's function [1]

$$I_o(z) = \frac{1}{2\pi i} \int_{P(z)} e^{z\zeta} g(\zeta) \, (\zeta + i)^{-1} \, d\zeta$$

where $g(\zeta)$ is the analytic function defined in the right half plane by (see [3], p. 91)

$$g(\zeta) = \exp\left\{\frac{1}{\pi} \int_0^\infty \log\left(\frac{t^{1/\alpha}}{t^{1/\alpha}-1}\ \frac{t^2-1}{t^2}\right) \frac{\zeta}{t^2+\zeta^2}\, dt\right\}$$

and where the path $P(z)$ comes from $\infty e^{-i\pi}$, goes counterclockwise around the origin outside the unit circle and out to $\infty e^{\pi i}$, if $\mathrm{Re}\, z \geq 0$. According to [1], $g(\zeta)$ is regular in the wedge defined by

$$\zeta = re^{i\theta\pi}, \ r > 0, \ -\tfrac{1}{2} - 2\alpha + \delta \leq \theta \leq \tfrac{1}{2} + 2\alpha - \delta$$

with arbitrary small $\delta > 0$ and tends there to 1 as $|\zeta|$ tends to ∞ Furthermore at the origin

$$g(\zeta) = \zeta^{\frac{1}{2\alpha} - 1} \ (1 + \gamma(\zeta))$$

where $\gamma(\zeta)$ is regular in ζ and continuous in ζ and α in $0 < \alpha \leq 1$, $|\zeta| < |\zeta_0|$ with a fixed ζ_0 independent of α. $I_0(z)$ is a bounded solution of the surface wave problem for angle $\pi\alpha$ between surface and bottom. As $z \to \infty$ in a sector $\delta \geq \arg z \geq -\alpha\pi + \delta$ with $0 < \delta < \alpha\pi$ we have the estimate

(1.1) $$I_0(z) = A e^{-iz} + B z^{-1/2\alpha} + o(z^{-1/2\alpha})$$

where A and B are continuous functions of α whose exact values do not matter except for $A \neq 0$.

For z on a ray $z = z\ e^{-i\pi\alpha + \pi i/2}$, we choose for $P(z)$ a path coming from $\infty e^{-\frac{3}{2}\pi i + \pi i \alpha}$ on a ray through the origin, avoiding and surrounding the unit circle in the positive sense and returning back on the same ray to $\infty e^{\pi i\alpha + \pi i/2}$. For the computation of $I_0(z)$ we may replace the portion of the path $P(z)$ traveled twice in opposite direction outside the unit circle by the same path traveled simply to $\infty e^{\pi i(\alpha + \frac{1}{2})}$

but with the integrand $g(\zeta)$ replaced by the integrand $g(\zeta) - g(\zeta e^{-2\pi i})$. Employing the same technique as in [1] we find for $|\zeta| > 1$ on $\zeta = |\zeta| e^{\pi i(\alpha + \frac{1}{2})}$

$$|g(\zeta) - g(\zeta e^{-2\pi i})|$$

$$= 2|\sin \pi/(2\alpha)| |\zeta|^{\frac{1}{2\alpha} - 1} (|\zeta| + 1)^{1/2} (|\zeta|^2 + 2 \cos \pi\alpha |\zeta| + 1)^{1/2}$$

$$\cdot (|\zeta| - 1)^{-1/2} (|\zeta|^{1/\alpha} + 1)^{-1/2} (|\zeta|^{2/\alpha} + 2 \cos(\pi/\alpha) |\zeta| + 1)^{-1/2}$$

by approximation of α through rationals of form $p/(2q)$. Hence

$$I_o'(z) = \frac{1}{2\pi i} \int_{P(z)} e^{z\zeta} g(\zeta) \frac{\zeta}{\zeta + i} d\zeta$$

$$= f_1(z) + f_2(z)$$

where $f_2(z) = (2\pi i)^{-1} \int_{2e^{\pi i(\alpha + \frac{1}{2})}}^{\infty e^{\pi i(\alpha + \frac{1}{2})}} e^{z\zeta} (g(\zeta) - g(\zeta e^{-2\pi i}) \frac{\zeta}{\zeta + i} d\zeta$,

and $f_1(z)$ is a regular function of z and continuous in α and z. Hence

$$\int_0^z |f_2(t)| |dt| \leq \int_0^{|z|} dt \int_2^{\infty} e^{-t\zeta} K \zeta^{-\frac{1}{\alpha}} d\zeta$$

$$\leq K \int_2^{\infty} (1 - e^{|z|\zeta}) \zeta^{-\frac{1}{\alpha} - 1} d\zeta \leq K\alpha \cdot 2^{-\frac{1}{\alpha}} \leq K\alpha$$

with K a constant independent of α. Thus it follows that the function $I_o(z)$ is of bounded variation on a segment of a ray of angle $(-\alpha + 1/2)\pi$, extending from the origin to any finite distance r, and that this total variation is uniformly bounded by some $M(r)$ for any closed set of

$0 < \alpha \leq 1$. Since $I_o(z)$ is real on the ray of angle $-\pi\alpha$, the total variation of $I_o(z)$ is bounded by the same bound on the ray of angle $(-\alpha -1/2)\pi$. By familiar reasoning, the total variation of $I_o(z)$ is hence also uniformly bounded on any ray of angle $\pi\theta$ in $-\alpha - 1/2 \leq \theta \leq -\alpha + 1/2$. From the functional equation satisfied by $I_o(z)$ follows the extension of this result to an arbitrary fixed interval for the angle $\pi\theta$ of the ray, and finally not only for rays through the origin, but likewise for circles through the origin whose radii are equal to a fixed number r.

The functions $J_k(z)$ introduced by R. S. Lehman [2], p. 104, for $k = 1, 2, \cdots$ can be extended to the value $k = 0$ by setting

$$(1.2) \qquad J_k(z) = \int_0^\infty I_o(t) \quad (z-t)^{-k/\alpha -1}\, dt, \qquad k = 0, 1, \cdots$$

with the path of integration determined by requiring it to leave z to the right if $\arg z = 0$. The integral is no longer absolutely convergent for $k = 0$ but its convergence follows from the asymptotic estimate (1.1) of $I_o(z)$. Computation yields

$$(1.3) \qquad \begin{aligned} J_k(z) &= A \int_0^\infty e^{-it}\, (z-t)^{-\beta -1}\, dt + o(1), \qquad \beta = k/\alpha, \\ &= 2\pi i\, e^{\beta\pi i/2}\, \Gamma^{-1}(1+\beta)\, A\, e^{-iz} + o(1) \end{aligned}$$

as $z \to \infty$ in $0 \geq \arg z > -\alpha\pi$. On the other hand, if $\delta > 0$ is small, then,

$$(1.4) \qquad |J_k(z)| \leq \text{const.}\, |z|^{-k/\alpha}, \text{ as } |z| \to \infty, \; -\delta \geq \arg z \geq -2\pi.$$

In view of (1.3) we introduce the functions $I_{-k}(z)$ defined by

$$(1.5) \qquad I_{-k}(z) = \Gamma(1 + k/\alpha)\, J_k(z) - e^{\pi i/(2\alpha)}\, \Gamma(1+(k-1)/\alpha)\, J_{k-1}(z); \; k=1,2,\ldots.$$

They satisfy the relations

(1.6) $| I_{-k}(z) | \leq \text{const} \, |z|^{-(k-1)/\alpha}, \quad | I'_{-k}(z) | \leq \text{const.} |z|^{-1-(k-1)/\alpha}$

as $|z| \to \infty$ in $0 \geq \arg z \geq -2\pi$.

For positive k the functions $I_k(z)$ are defined ([2], p. 100), as the fractional integrals of $I_o(z)$,

(1.7) $$I_k(z) = \int_o^z I_o(t)\,(z-t)^{k/\alpha - 1}\, dt \cdot \Gamma^{-1}(k/\alpha), \qquad k \geq 1.$$

The functions I_k, J_k are continuous functions of z and α for $z \neq 0$, $0 < \alpha \leq 1$. Furthermore

$$I_o(0) = 1; \quad I_k(0) = 0,\ k > 0; \quad J_o(z) = \log z + \text{const} + o(1)$$

as $z \to 0$.

Of great consequence is the formula

(1.8) $$J_o(z) - J_o(ze^{-2\pi i}) = + 2\pi i I_o(z)$$

which is a direct consequence of the definition of $J_o(z)$.

2. <u>Bilinear Invariant</u>.

For two solutions f and g of (E) we form the invariant (see [2], p. 109)

$$Q[f',g] = 2\int_{z\varepsilon}^{z} f'(t)\, g(t)\, dt + (f(t) + f(t\varepsilon))\,(g(t\varepsilon) - g(t)).$$

Q is independent of the choice of z and

(2.1) $$Q[f',g] = -Q[g',f].$$

Evaluating the invariant for $z \to 0$, we find in particular

$$Q[I_o', J_o] = -Q[J_o', I_o] = -4\pi i\alpha.$$

Furthermore $Q[I_k', J_o] = 0$, $k = 1, 2, \cdots$, and $Q[J_o', J_o] = 0$. For $\alpha < 1$, we have $Q[J_1', J_o] = 0$ since by (1.2), for $|z|$ large, $0 \geq \arg z \geq -2\pi\alpha$, certainly $|J_1'(z)| \leq |z|^{-1/\alpha}$ const and $|J_o(z)|$ remains bounded, so that the invariant, evaluated for $z \to \infty$ tends to 0. Consequently $Q[J_1', J_o] = 0$ also for $\alpha = 1$, since the invariant may be formed for $|z| = 1$, and on $|z| = 1$, J_k and J_k' are continuous in z and α. Thus we find

$$Q[J_o', I_{-1}] = 0,$$

and by a similar, even simpler argument

$$Q[J_o', I_{-k}] = 0, \qquad k = 1, 2, \cdots.$$

Furthermore also by (1.6) and (2.1)

$$Q[I_{-j}', I_{-k}] = 0 \qquad j, k = 1, 2, \cdots.$$

The essential usefulness of the functions I_{-k} is based on the formula of [2] Lemma 11.3 which states that for $k > \nu \geq 0$

$$Q[J_k', I_\nu] = e^{(k-\nu)\pi i/2\alpha}\, Q[J_k', I_k]$$

$$= e^{(k-\nu)\pi i/2\alpha}\, 4\pi i \alpha \Gamma^{-1}(1 + k/\alpha).$$

Hence, by (1.5),

$$Q[I_{-k}', I_\nu] = 0 \qquad \text{for } k > \nu.$$

Moreover

$$Q[I_{-k}', I_k] = 4\pi i \alpha,$$

since by [2] Lemma 11.1, $Q[J'_k, I_\nu] = 0$ for $\nu > k$. Likewise

$$Q[I'_{-k}, I_\nu] = 0, \qquad \nu > k.$$

Set for brevity's sake

$$Q[f', g] \,/\, 4\pi i\alpha = [f,g].$$

Then we have proven

$$(2.1) \qquad [f,g] = -[g,f]$$

and

$$(2.2) \qquad [I_k, I_j] = \begin{cases} 0, & k = j \neq 0 \quad \text{or } k = j = 0, \\ 1, & k + j = 0, \quad k < j, \\ -1, & k + j = 0, \quad k > j, \end{cases}$$

and

$$[I_k, J_o] = \begin{cases} -1, & k = 0, \\ 0, & k \neq 0. \end{cases}$$

3. Further properties of I_{-k}.

As pointed out in [2], p. 100, the $I_k(z)$ are the successive fractional integrals of $I_o(z)$, i.e. for $k \geq 1$,

$$I_k(z) = D^{-\frac{k}{\alpha}} I_o(z) = \Gamma^{-1}(k/\alpha) \int_o^z (z - t)^{\frac{k}{\alpha} - 1} I_o(t)\, dt.$$

A similar relation holds for the I_{-k} with negative index. But the definition of the operator $D^{-\frac{k}{\alpha}}$ for functions which become infinite at $z = 0$ must be changed. Observe that if $F(z)$ satisfies the estimate $|F(z)| \leq \text{const}\, |z|^{-\lambda}$, where $\lambda > 0$, then the definition (with positive integral j)

$$D^{-\frac{j}{\alpha}} F(Z) = \Gamma^{-1}(j/\alpha) \int_Z^\infty (z - Z)^{j/\alpha - 1} F(z)\, dz$$

converges for $0 < j/\alpha < \lambda$. Now let $0 \leq t < z$. Calculation gives

(3.1) $$D^{-j/\alpha} (z-t)^{-\lambda} \Gamma(1 + \lambda) = \Gamma(\lambda - j/\alpha)(z - t)^{-\lambda + j/\alpha}$$

a formula first valid for $t < z$ and then for arbitrary t, z by analytic extension. Hence

$$D^{-j/\alpha} \Gamma(1 + k/\alpha) J_k(z) = \Gamma(1 + (k-j)/\alpha) J_{k-j}(z), \quad 0 < j < k,$$

and consequently by (1.5)

(3.2) $$D^{-j/\alpha} I_{-k}(z) = I_{-(k-j)}(z), \qquad 1 \leq j \leq k - 1$$

4. r-<u>solutions</u>.

The equation

$$u'(z) - u'(\varepsilon z) + i(u(z) + u(\varepsilon z)) = 0$$

heretofore was always interpreted as a relation affecting functions of a complex variable, regular within some two-dimensional domain of the variable z. But if we fix attention on a single circumference, say $|z| = r$, it still makes sense to talk of a solution u of the equation, defined only on $|z| = r$, regardless of whether or not u can be extended into a neighborhood as a regular function. Such solution will be called an r-solution.

If the values of an r-solution $u(z)$ are known only on $0 \geq \arg z \geq -2\pi\alpha$ and have a continuous derivative there, then the equation permits to define $u(z)$ as r-solution on the arc $-2\pi\alpha \geq \arg z \geq -4\pi\alpha$, so that $u(z)$ remains continuous at $-2\pi\alpha$, and its derivative is continuous in $-2\pi\alpha \geq \arg z \geq -4\pi\alpha$, although it may have different limit values upon approach to $z = -2\pi\alpha$ from the inside of the two arcs meeting there. This process can be re-

peated so as to yield, starting with an arbitrary function on $|z| = r$, $0 \geq \arg z \geq -2\pi\alpha$ of class C', an r-solution $u(z)$ defined on the infinitely often wound circumference $|z| = r$ of the logarithmic Riemann surface; and $u(z)$ will have a derivative continuous on every partial arc $2\pi\nu\alpha \geq \arg z \geq 2\pi(\nu-1)\alpha$, $\nu = \cdots, -1, 0, 1, \cdots$. It is our purpose to show the developability of r-solutions in series of $J_0(z)$ and $I_k(z)$, $k = 0, \pm 1, \pm 2, \cdots$.

5. <u>The rational case</u>.

Assume now that α equals a reduced fraction $p/(2q)$ with even denominator, $0 < \alpha = p/(2q) < 1$. Let $F(z)$ be an r-solution, continuous on $|z| = r$, and with derivatives up to the order q continuous within each arc $2\pi\nu\alpha \leq \arg z \leq 2\pi(\nu + 1)\alpha$. Since by (E)

$$(D + i)\, F(z) = (D - i)\, F(\varepsilon z), \qquad \varepsilon = e^{-\pi i p/q}$$

we find (see also [3]) with

$$\varphi(D) = \prod_{\nu=0}^{q-1} (D + i\varepsilon^{\nu})$$

since $\varepsilon^q = -1$, that

$$\varphi(D)\, F(z) = \prod_{\nu=1}^{q} (D + i\varepsilon^{\nu})\, F(\varepsilon z) = -\prod_{\nu=0}^{q-1} (D\varepsilon^{-1} + i\varepsilon^{\nu})\, F(\varepsilon z)$$

$$= -\varphi(D\varepsilon^{-1})\, F(\varepsilon z).$$

Hence

$$\varphi(D\varepsilon^{-1})\, F(\varepsilon z) = -\varphi(D\varepsilon^{-2})\, F(\varepsilon^2 z), \cdots$$

and finally

$$\varphi(D)\, F(z) = \varphi(D)\, F(z\varepsilon^{2q}) = \varphi(D)\, F(ze^{-2\pi i p}).$$

Hence the difference $\delta(z) = F(z) - F(ze^{-2\pi i p})$ satisfies

$$\varphi(D)\,\delta(z) = 0$$

and $\delta(z)$ is a continuous r-solution. We find accordingly

$$\delta(z) = \sum_{0}^{q-1} h_\nu \, e^{-i\varepsilon^\nu z}$$

and substitution in (E) yields the further information

$$h_\nu = h_{\nu-1}\,(-i\varepsilon^\nu - i)\,/\,(-i\varepsilon^\nu + i), \qquad \nu = 1, \dots, q-1$$

which shows $\delta(z)$ to be determined but for a constant factor.

In particular consider the difference $J_0(z) - J_0(z\,e^{-2\pi ip})$. By (1.8) we see that

$$\delta(z) = J_0(z) - J_0(z\,e^{-2\pi ip}) = 2\pi i(I_0(z) + I_0(z\,e^{-2\pi i}) + \dots + I_0(ze^{-2\pi i(p-1)}))$$

and here right hand is not equal to the constant zero as it tends to $2\pi i\,p$ as $z \to 0$. It follows that we may subtract from our r-solution $F(z)$ a multiple $c\,J_0(z)$ of $J_0(z)$ so that $F(z) - c\,J_0(z)$ is periodic as arg z is replaced by arg $z + 2\pi p$. It thus appears that for the purpose of developability of F in terms of J_0 and I_k we may assume that

$$F(z) = F(z\,e^{-2\pi ip}). \tag{5.1}$$

Accordingly, with $F(z)$ on $|z| = r$ is associated the Laurent-power series

$$F(z) \sim \sum_{-\infty}^{\infty} f_\nu z^{\nu/p} \tag{5.2}$$

with the coefficients

$$f_\nu = \frac{1}{2\pi ip}\int F(z)\,z^{-\nu/p\,-1}\,dz$$

where the integral is extended over the circumference wound p times.

The familiar theory of Fourier series asserts that any function $F(z)$ with (5.1) which has piecewise continuous derivatives possesses a convergent Laurent power series and equals it. Thus a periodic r-solution $u(z)$ equals its convergent Laurent-power series. The equation (E) translates into a relation between the coefficients f_ν of u

(5.3) $\quad f_{\nu+p}(1 - e^{-\pi i(\nu+p)/q})(\nu+p) + ipf_\nu\,(1 + e^{-\pi i\nu/q}) = 0.$

Suppose an index $\nu + p$ to be an even multiple $2n\,q$ of q. Then $f_\nu = f_{2nq-p} = 0$, and furthermore $f_{2nq-2p} = f_{2nq-3p} = \cdots = f_{2nq-(q-1)p} = 0.$

Accordingly there are solutions of the form

$$z^{2nq/p}\ \ p_n(z), \qquad\qquad n = 0, \pm 1, \cdots$$

where $p_n(z)$ is a polynomial of degree q, and $u(z)$ is a sum of such special solutions.

Observation.

We have seen that for an r-solution ω with continuous derivatives of order q on each arc $2\pi\alpha\nu \leq \arg z \leq 2\pi\,(\nu+1)\alpha$, $\nu = 0, \pm 1, \pm 2, \cdots$ there is a suitable constant c such that $\omega - cJ_o$ satisfies the periodicity condition (5.1). It is easily established that c depends on ω in a continuous way. Now consider an r-solution u possessing only a continuous first derivative on every arc defined above. Let us approximate u and u' on the arc $0 \geq \arg z \geq -2\pi\alpha$ by functions ω and ω' where ω possesses continuous derivatives up to order q on this arc. (This is possible by Weierstrass' Theorem.) We extend ω into r-solutions by the equation (E). Observe that the smallness of $|u - \omega|$ and $|u' - \omega'|$ on one arc implies smallness on an adjacent arc for the same quantities. The periodic r-solutions $\omega - c\,J_o$ have as limits evidently a periodic

r-solution $u - \lim c J_o$.

Extension of Lehman's theorem [2], 12.1.

We shall now give a generalization of the developability of solutions which are regular near the origin and continuous at $z = 0$ to solutions $G(z)$ which are regular near the origin and which tend to ∞ less than a negative power of z as $z \to 0$. We need only a special form here; greater generality will follow from our general development theorem.

Lemma 1: Let $\alpha = p/(2q)$. Let $G(z)$ be a solution of (E), regular for $z \neq 0$, and $|G(z)| \leq |z|^{-m/\alpha}\cdot$const. for some integer $m > 0$ for bounded $|\arg z|$. Then $G(z)$ can be developed in a series

$$G(z) = \gamma J_o(z) + \sum_{-m}^{\infty} c_k I_k(z)$$

which converges absolutely and uniformly in every circle $|z| \leq R$.

Proof: If $G(z) - c J_o(z)$ is developed in a Laurent-power series in $z^{1/p}$, it must become of the form $\sum_{-m}^{\infty} a_\nu z^{2\nu q/p} p_\nu(z)$. Now by [2], (10.9), $J_k(z)$, $k \geq 1$ and consequently $I_{-k}(z)$ start with the nonvanishing term of highest order $z^{-k/\alpha}$. Accordingly coefficients c_{-m} and c' can be assigned so that the Laurent-power series of $G(z) - c_{-m} I_{-m}(z) - c' J_o(z)$ starts with terms of order no higher than $z^{(-m+1)/\alpha}$.

Thus continuing we arrive at successive coefficients c_ν for negative ν, such that $G(z) - c_{-m} I_{-m}(z) - \cdots - c_{-1} I_{-1}(z)$ is certainly $o(z^{-2q/p})$ near $z = 0$. If we subtract further a suitable multiple of $J_o(z)$ then the result will be periodic of period $2\pi p$ in arg z, and equal a power series without negative powers of $z^{1/p}$, thus be developable according to Lehman's Theorem. The lemma now follows.

Lemma 2: Let $\nu \geq (p+1)/2$. Define as $\pi_{-\nu}$ the finite power sum $z^{-\nu 2q/p} P_{-\nu}(z) = -\Gamma(\nu/\alpha) z^{-\nu 2q/p} + \ldots$ which is that solution which by [2], (10.9) is generated by the term of highest order of infinity in $I_{-\nu}(z)$. (Note that $\pi_{-\nu}$ contains only negative powers.) There are $(p-1)/2$ constants $a_1, \ldots, a_{(p-1)/2}$, independent of ν , and such that

$$(5.4) \qquad \pi_{-\nu} = I_{-\nu} + a_1 I_{-\nu+1} + \ldots + a_{(p-1)/2} I_{-\nu+(p-1)/2} \ .$$

Proof: By Lemma 1, we have

$$\pi_{-\nu} = I_{-\nu} + a_1 I_{-\nu+1} + \ldots + a_{(p-1)/2} I_{-\nu+(p-1)/2} +$$

$$+ cJ_o + \sum_{-\nu+(p+1)/2}^{\infty} b_\mu I_\mu(z) \ .$$

Now, since in $0 \geq \arg z \geq -2\pi\alpha$ the functions I_k for $k \geq 0$ are for large $|z|$ bounded by $|z|^{k/\alpha}$ const , we must have

$$[I_o, \pi_{-\nu}] = [I_1, \pi_{-\nu}] = \ldots = [I_{\nu-(p+1)2}, \pi_{-\nu}] = 0 \ ,$$

or $c = b_{-1} = \ldots = b_{-\nu+(p+1)/2} = 0$. For $\pi_{-\nu}$ is at ∞ of order $z^{-\nu . 2q/p+q}$, $I_{\nu-(p+1)/2}(z)$ of order $z^{(+\nu-(p+1)/2).2q/p}$, so that $[I_{+\nu-(p+1)/2}, \pi_{-\nu}]$ is of order $z^{q-(p+1)q/p} = z^{-q/p}$, hence vanishes. Likewise $[J_o, \pi_{-\nu}] = 0$ whence $b_o = 0$. On the other hand, $[I_{-n}, \pi_{-\nu}] = 0$ for $n \geq 1$ since at ∞ , the I_{-n} are $o(1)$. Hence $b_n = 0$ for $n > 0$. That the a_μ are independent of ν follows from the fact that for $\nu > (p+1)/2$ we may apply to both sides of (5.4) the operator $D^{-2q/p}$. Right hand thereby goes into the expression (see (3.2))

$$I_{-\nu+1} + a_1 I_{-\nu+2} + \dots + a_{(p-1)/2} I_{-\nu+1+(p-1)/2}$$

hence into a solution of (E). Thus left hand goes into a solution, and since each power of z goes necessarily into another power of z with exponent larger by $2q/p$, $\Pi_{-\nu+1}$ is necessarily generated from $I_{-\nu+1}$ as $\Pi_{-\nu}$ from $I_{-\nu}$.

A similar lemma holds for the solution $z^{2kq/p}\, p_k(z) = \Pi_k$ with which the development at the origin of $I_k(z)$ begins, for $k \geq 0$.

Lemma 3: For $k \geq 0$, we have

$$(5.5) \qquad \Pi_k = I_k + a_1' I_{k+1} + \dots + a'_{(p+1)/2} I_{k+(p+1)/2} ,$$

where the coefficients a_ν' are independent of k.

Proof: By Lehman's theorem [2] 12.1,

$$\Pi_0 = I_0 + a_1' I_1 + \dots .$$

Now the highest degree occurring in Π_0 is q. Hence $[I_{-(p+3)/2}, \Pi_0] = 0$ since this expression, by (1.6), is of order $|z|^{-(p+1)q/p+q} = |z|^{-q/p}$ as $z \to \infty$. Similarly $[I_{-\nu}, \Pi_0] = 0$ for $\nu > (p+3)/2$. Thus all coefficients a_ν' vanish for $\nu > (p+1)/2$. Application of the operator $D^{-2q/p}$ changes Π_k into Π_{k+1} if $k \geq 0$, and I_k into I_{k+1}, q.e.d.

For the solutions $\Pi_{-(p-1)/2}, \dots, \Pi_{-1}$ we do not give a more precise development, except for remarking that they admit of the general development of Lemma 1.

The question now arises whether the given Laurent-power development of an r-solution $u(z)$ with $u(z) = u(ze^{2\pi i p})$ can be rewritten as a convergent development in the functions J_0 and I_k, $k = 0, \pm 1, \pm 2, \dots$.

First of all, if the Laurent-power series of u converges absolutely, a reordering permits us to write

$$u = cJ_0 + \sum_{-\infty}^{\infty} c_\nu \Pi_\nu$$

where the c_ν and the coefficients f_ν of u are related by

$$c_\nu \Gamma^{-1}(1 + \nu/\alpha) = f_{2\nu q} = f'_\nu , \quad \nu \geq 0$$

$$- c_\nu \Gamma(-\nu/\alpha) = f_{2\nu q} = f'_\nu , \quad \nu < 0.$$

Hence

$$u - cJ_0 - \sum_{-(p-1)/2}^{-1} c_\nu \Pi_\nu = \sum_0^{\infty} c_\nu (I_\nu + a'_1 I_{\nu+1} + \dots + a'_{(p+1)/2} I_{\nu + \frac{p+1}{2}})$$

$$+ \sum_{-\infty}^{-(p+1)/2} c_\nu (I_\nu + a_1 I_{\nu+1} + \dots + a_{(p-1)/2} I_{\nu + \frac{p-1}{2}}) .$$

Now it will be seen that on $|z| = r$

(6.2) $$|I_\nu| \leq M r^{\frac{\nu}{\alpha}} \Gamma^{-1}(1 + \nu/\alpha), \quad \nu \geq 0,$$

(6.1) $$|I_\nu| \leq M r^{\frac{\nu}{\alpha}} \Gamma(- \nu/\alpha), \quad \nu \leq -3.$$

Thus

$$\sum_0^{\infty} |c_\nu I_\nu| \ll M \sum_0^{\infty} |f'_\nu| \, r^{\nu/\alpha},$$

$$\sum_0^{\infty} |c_\nu I_{\nu+j}| \ll M \sum_0^{\infty} |f'_\nu| \, r^{(\nu+j)/\alpha} \, \Gamma(1 + \nu/\alpha) \Gamma^{-1}(1 + (j + \nu)/\alpha)$$

$$\ll M \sum_0^{\infty} |f'_\nu| \, r^{(\nu+j)/\alpha} , \quad j \geq 1,$$

and

$$\sum_{-\infty}^{-(p+1)/2} |c_\nu I_\nu| \ll M \sum_{-\infty}^{-(p+1)/2} r^{\nu/\alpha} |f'_\nu| ,$$

$$\sum_{-\infty}^{-(p+1)/2} |c_\nu I_{\nu+j}| \ll M \sum_{-\infty}^{-(p+1)/2} r^{(\nu+j)/\alpha} |f'_\nu| \Gamma^{-1}(\tfrac{-\nu}{\alpha}) \Gamma(\tfrac{-\nu+j}{\alpha})$$

$$\ll M \sum_{-\infty}^{-(p+1)/2} r^{(\nu+j)/\alpha} |f'_\nu| , \quad 1 \leq j \leq (p-1)/2.$$

Furthermore the finite sum $\sum_{-(p-1)/2}^{-1} c_\nu \Pi_\nu$ is a solution regular in the whole plane for $z \neq 0$, hence can itself be developed into an absolutely convergent series in J_o and I_k by our Lemma 1.

Thus if the r-solution $u(z)$ has an absolutely convergent Laurent-power series then it also admits of an absolutely convergent development into a series in J_o and I_k.

It is desirable to state sufficient conditions relating to the behavior of u on one arc, say $0 \geq \arg z \geq -2\pi\alpha$, which insure the absolute and uniform convergence of the Laurent-power series of the r-solution $u - cJ_o$. It follows from familiar facts that existence and boundedness of the second derivative $u''(z)$ on $|z| = r$, $0 \geq \arg z \geq -2\pi\alpha$ yield absolute and uniform convergence of the corresponding Laurent-power series, hence also of the development of u in a series in J_o and I_k.

Definition: An admissible r-solution u is a continuous r-solution with bounded second derivatives in $0 \geq \arg z \geq -2\pi\alpha$.

6. General α.

The estimate (6.2) follows from the definition (1.7) of I_k for $k > 0$ with M an upper bound for the module of $I_o(z)$ in $|z| \leq r$, and $0 \geq \arg z \geq -2\pi\alpha$. The estimate (6.1) is obtained in several steps: First observe that the definition of $J_k(z)$ permits us to write for $k \geq 2$,

$$J_k(z) = -\frac{\alpha}{k} z^{-k/\alpha} - \frac{\alpha}{k} \int_0^\infty (z-t)^{-k/\alpha} I_o'(t)dt .$$

Now on any circle of radius r passing through the origin and of center r $\int |I_o'(t)||dt|$ remains uniformly bounded for $\alpha_o \leq \alpha \leq 1$ by a number $M(r)$ independent of α. If $0 \geq \arg z \geq -2\pi$, one of the semicircles starting at the origin will have distance r from z on $|z| = r$; hence, the last integral over the semicircle is absolutely no more than the bound $r^{-k/\alpha}M(r)$. On the remainder of the path of integration, $|I_o'(t)|$ remains bounded, by (1.1), hence again, $\left|\int_{2r}^\infty\right| \leq r^{-k/\alpha}M(r)$, with suitable $M(r)$.

Thus for $k \geq 2$, $|J_k(z)| \leq \frac{\alpha}{k} r^{-k/\alpha} M(r)$, $|z| = r$, $0 \geq \arg z \geq -2\pi\alpha$ and, by (1.5), for $k \leq -3$

$$|I_k(z)| \leq \Gamma\left(\frac{-k}{\alpha}\right)r^{-k/\alpha}M(r) + \Gamma\left(\frac{-k+1}{\alpha}\right)r^{-k/\alpha}M(r) ,$$

or

$$(6.1) \qquad |I_k(z)| \leq \Gamma\left(\frac{-k}{\alpha}\right)r^{-k/\alpha}M(r) , \quad \text{on } |z| = r , \ 0 \geq \arg z \geq -2\pi\alpha , \quad k \leq -3 .$$

Returning to the case $\alpha = p/(2q)$, let u be an admissible r-solution whence u equals the uniform limit on $|z| = r$, $0 \geq \arg z > -2\pi\alpha$,

$$u(z) = cJ_o(z) + \sum_{-\infty}^{\infty} c_k I_k(z) .$$

Forming the invariant $[I_k,u]$ on the circle $|z| = r$, we obtain therefore

(6.3)
$$c_k = [I_{-k},u] , \qquad k > 1$$
$$c_k = -[I_k,u] , \qquad k < 1$$
$$c_o = [J_o,u] \text{ and } c = -[I_o,u] .$$

We now restrict α to a fixed interval $0 < \alpha_o \leq \alpha \leq 1$ and estimate the magnitude of the coefficients as $|k| \longrightarrow \infty$.

For $k = -j < 0$

$$4\pi i\alpha c_k = 4\pi i\alpha[u,I_j'] = 2\int_{r\varepsilon}^{r} u'(t)I_j(t)dt + (u(r) + u(r\varepsilon))(I_j(r\varepsilon) - I_j(r))$$

$$= 2u'(r)\int_{r\varepsilon}^{r} I_j(t)dt + (u(r) + u(r\varepsilon))(I_j(r\varepsilon) - I_j(r))$$

$$- 2\int_{r\varepsilon}^{r} u''(t)dt \int_{r\varepsilon}^{t} I_j(t')dt' .$$

Now

$$I_j(r) - I_j(r\varepsilon) = \Gamma^{-1}(j/\alpha)\left(\int_o^r (r-t)^{j/\alpha -1} I_o(t)dt - \int_o^{r\varepsilon} (r\varepsilon -t)^{j/\alpha -1} I_o(t)dt\right)$$

$$= \Gamma^{-1}(j/\alpha)\int_o^r (r-t)^{j/\alpha -1}(I_o(t) - I_o(\varepsilon t))dt$$

$$= -i\Gamma^{-1}(j/\alpha+1)\int_o^r (r-t)^{j/\alpha} (I_o(t) + I_o(\varepsilon t))dt .$$

Thus

$$|I_j(r) - I_j(r\varepsilon)| \leq \Gamma^{-1}(2 + j/\alpha)r^{j/\alpha} M(r) \quad ,$$

since $I_o(z)$ depends continuously on α. Next

$$\int_r^z I_j(t)dt = \int_{r\varepsilon}^z dt \int_o^t (t - t')^{j/\alpha - 1} I_o(t')dt'\Gamma^{-1}(j/\alpha)$$

$$= -\int_{r\varepsilon}^z dt \int_o^t (t - t')^{j/\alpha} I_o'(t')dt'\Gamma^{-1}(1 + j/\alpha)$$

$$+ ((r\varepsilon)^{j/\alpha + 1} - z^{j/\alpha + 1})\Gamma^{-1}(2 + j/\alpha) \quad .$$

Therefore,

$$\left| \int_r^z I_j(t)dt \right| \leq \Gamma^{-1}(2 + j/\alpha)M(r)r^{j/\alpha} \quad ,$$

since $\int_o^z |dI_o(t)|$ depends continuously on α and z for $\alpha_o \leq \alpha \leq 1$.
Hence for $k = -j < 0$

$$|c_k| \leq \Gamma^{-1}(2 + j/\alpha)r^{j/\alpha} M(r)(U + U' + U'')$$

where U , U' , U'' are, respectively, $\max|u|$, $\max|u'|$, $\max|u''|$ in $0 \geq \arg z \geq -2\pi\alpha$. Accordingly, by (6.1),

$$(6.5) \qquad |c_k I_k| \leq \frac{(U + U' + U'')M(r)}{|k|(|k|+\alpha)} \, , \; k < -3 \quad .$$

Now for $k > 2$,

$$- 4\pi i \alpha\, c_k = -[I_{-k},u] = +[u,I_{-k}]$$

$$= 2u'(r)\int_{r\varepsilon}^{r} I_{-k}(t)dt + (u(r) + u(r\varepsilon))(I_{-k}(r\varepsilon) - I_{-k}(r))$$

$$- 2\int_{r\varepsilon}^{r} u''(t)dt \int_{r\varepsilon}^{t} I_{-k}(t')dt' \; ;$$

$$J_k(r) - J_k(r\varepsilon) = \int_0^{\infty} (r - t)^{-k/\alpha - 1} I_o(t)dt - \int_0^{\infty} (r\varepsilon - t)^{-k/\alpha - 1} I_o(t)dt$$

$$= \int_0^{\infty} (r - t)^{-k/\alpha} (I_o(t) + I_o(\varepsilon t))dt \frac{-\alpha i}{k} ,$$

$$|J_k(r) - J_k(r\varepsilon)| \leq M(r)\frac{\alpha}{k}\,\frac{\alpha}{k-\alpha}\, r^{-k/\alpha} ,$$

$$|I_{-k}(r) - I_{-k}(r\varepsilon)| \leq r^{-k/\alpha}\, M(r)\Gamma(-1 + k/\alpha) , \qquad k \geq 3$$

Moreover

$$\int_{r\varepsilon}^{z} J_k(t)dt = + \frac{\alpha}{k} \int_0^{\infty} I_o(t)((z - t)^{-k/\alpha} - (r\varepsilon - t)^{-k/\alpha})dt$$

and as above,

$$\left| \int_{r\varepsilon}^{z} J_k(t)dt \right| \leq + \frac{\alpha}{k}\, \frac{\alpha}{k-\alpha}\, r^{-k/\alpha}\, M(r) ,$$

$$\left| \int_{r}^{z} I_{-k}(t)dt \right| \leq \Gamma(\frac{+k}{\alpha} - 1)\, r^{-k/\alpha}\, M(r) ,$$

$$|c_k| \leq \Gamma(\frac{+k}{\alpha} - 1)\, r^{-k/\alpha}\, M(r)(U + U' + U'') .$$

(6.4) $$|c_k I_k| \leq (U + U' + U'') \frac{M(r)\, r^{-k/\alpha}}{k(k - \alpha)}, \quad k \geq 3$$

Thus $\sum_{-\infty}^{\infty} c_k I_k$ converges absolutely and uniformly for all $\alpha = \frac{p}{2q}$ in $0 < \alpha_o \leq \alpha \leq 1$ and all admissible r-solutions u with the same upper bounds U, U', U'' for $|u|$, $|u'|$, $|u''|$ in the respective arcs $0 \geq \arg z \geq -2\pi\alpha$ of $|z| = r$. It is easily seen that the convergence is absolute and uniform toward $u(z)$ for all z with bounded $|\arg z|$, $|z| = r$, since the coefficients are obtained as invariants of two solutions independently of r, and the estimates of an r-solution can be appropriately transferred to adjacent arcs. Let now α be an arbitrary number in $0 < \alpha \leq 1$. We enclose it in an interval $\alpha_o < \alpha \leq 1$, $\alpha_o > 0$. Let there be given, in $0 \geq \arg z > -2\pi\alpha$ on $|z| = r$, a function $u(z)$ continuous with its first derivative and with a bounded second derivative of the respective bounds U, $U'/2$, $U''/2$. We approximate α from below by a sequence of α_ν of form $p/(2q)$, and for each of these we transfer the given function u from its arc to the arc $0 \geq \arg z \geq -2\pi\alpha_\nu$ by subjecting the transferred function $u_\nu(z)$ to the rule

$$u_\nu(z) = u(re^{i(\arg z)\alpha/\alpha_\nu}).$$

Then $u_\nu(z)$ has on its arc the bounds U, U', U'' for $|u_\nu|$, $|u_\nu'|$, $|u_\nu''|$. Each u_ν as well as u generates an r-solution belonging to the angles $2\pi\alpha_\nu$ and $2\pi\alpha$, respectively, and we have $u(z) = \lim u_\nu(z)$. Furthermore the developments of $u_\nu(z)$ in series of J_o and I_k have by (6.4) and (6.5) the same majorization by an absolutely convergent series, while

the terms individually converge as $\alpha_\nu \longrightarrow \alpha$, since both c_k and I_k converge. It follows that the given $u(z)$ can be developed in a series in J_0 and I_k belonging to the α associated with u, with the coefficients given by (6.3). Hence the

Theorem: A continuous r-solution $u(z)$ with bounded second derivatives on $0 \geq \arg z \geq -2\pi\alpha$ can be developed in an absolutely and uniformly convergent series in J_0 and I_k on $|z| = r$ for bounded $|\arg z|$.

The two most important special cases are Lehman's theorem in which are considered regular solutions in $|z| \leq R$, and continuous at the origin, and the case of solutions regular for $|z| \geq r$, which in $0 \geq \arg z \geq -2\pi\alpha$ tend to zero as $z \rightarrow \infty$. In Lehman's case J_0 and I_k with negative k are absent from the development, in the other case the functions J_0 and I_k for $k \geq 0$. The connection between our Theorem and these two special cases is made by placing circles $|z| = r$ into the domain of regularity, applying the theorem and observing that the coefficients, being invariants of two solutions, cannot depend on r. There is an important difference in the two results, however. In Lehman's case, the convergence is uniform for bounded $|\arg z|$ and $|z| \leq R$. In the other case, however, all that can be asserted is that the convergence is uniform on every ring domain $R \leq |z| \leq R' < \infty$ with bounded $|\arg z|$.

The essential difference between the two cases is appreciated when it is noted that a solution regular for small $|z|$ and continuous as $z \rightarrow 0$ in <u>some</u> sector is continuous at the origin in <u>any</u> sector. On the other hand, the function $J_0(z)$ does not tend to zero as $z \rightarrow \infty$ and $\arg z = 0$, but does tend to zero as $|z| \rightarrow \infty$ and $-2\pi\alpha \geq \arg z \geq -4\pi\alpha$, if α is sufficiently small.

In order to see that for regular solutions $u(z)$ tending to 0 as $z \to \infty$ in $0 \geq \arg z \geq -2\pi\alpha$, the coefficients of $I_k, k \geq 0$ and of J_c are all zero observe that $|J_0'(z) - 2\pi Ae^{-iz}| \leq \text{const } |z|^{-1}$ as $z \to \infty$ in $0 \geq \arg z \geq -\alpha\pi/2$ and $|J_0'| \leq |z|^{-1}$ constant elsewhere. Hence $[J_c, u] = 0$. For $k \geq 1$, (1.6) yields $[I_{-k}, u] = 0$. Furthermore, the coefficient of J_0 must be zero since $u \to 0$ as $z \to +\infty$ but $J_0 \not\to 0$ while all I_k, $k \leq 0$, do.

A corrollary of the development theorem is the "completeness" relation. Let u and ω be admissible r-solutions, their coefficients c, c_k and γ, γ_k, respectively. Then

$$[u, \omega] = c\gamma_0 - c_0\gamma + \sum_1^\infty (c_{-k}\gamma_k - c_k\gamma_{-k}) .$$

For the proof, assume first that all but finitely many of the c_k are zero. Then the relation is trivial. Thus if we put

$$u_n = cJ + \sum_{-n}^{n} c_k I_k ,$$

we find

$$[u, \omega] = -[\omega, u] = -\lim_{n \to \infty} [\omega, u_n]$$

$$= \lim_{n \to \infty} \left\{ -c\gamma_0 + \gamma c_0 + \sum_1^n (-\gamma_{-k}c_k + \gamma_k c_{-k}) \right\} .$$

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